Relations Between Denominator Assigning Proper Feedback Compensators and Pole Assignment by State Feedback

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***Abstract*— We examine relations between denominator assigning proper compensators in the feedback path of linear, time invariant (LTI) multivariable systems, described by square strictly proper transfer function matrices, and pole assignment by state variable feedback. Through these results we establish conditions for the existence and computation of such compensators.**

I. INTRODUCTION

Let Σ be a linear, time invariant (LTI), stabilizable multivariable system characterized by an input-output *strictly proper* $p \times m$ transfer function matrix $P(s)$ and let $\Sigma := (A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n})$ be a *minimal state space realization* of a $P(s)$. It is well known that if the state vector $x(t)$ of Σ is fully accessible to measurements, then the poles of Σ (eigenvalues of A) can be reassigned arbitrarily in the open left half of the complex plane via state feedback through a state feedback matrix $F \in \mathbb{R}^{m\times n}$ giving rise to a closed loop transfer function matrix $P_F(s)$. If the state vector is not fully accessible, then a asymptotic state observer can be used in order to construct an estimate $\hat{x}(t)$ of the state vector $x(t)$. In this paper we deal with the problem of what can be done with respect to pole assignment if the state vector $x(t)$ of Σ is not fully accessible and the use of an observer is not desirable. The question we pose is the following. Under what conditions on the strictly proper "plant" $P(s)$ there exists a dynamic feedback *proper* compensator $C(s)$ such that, if employed in the feedback path of $P(s)$, as indicated in figure 1, the transfer function matrix $P_C(s)$ of the closed loop system Σ_c is equal to the transfer function matrix $P_F(s)$ obtained by state feedback. If such a proper compensator $C(s)$ exists, it mimics a state observer without making use of the reference input as it is required in the observer design. We firstly examine relations between denominator assigning proper compensators $C(s)$ in the feedback path of LTI multivariable systems, described by strictly proper transfer function matrices, and pole assignment by state variable feedback. Through these results we establish conditions for the existence and computation of proper compensators $C(s)$ resulting to a closed loop feedback system Σ_c, which is internally stable and is characterized by a closed loop transfer function matrix $P_C(s)$ that can be obtained by state feedback on a minimal state space realization of $P(s)$. It is shown that if the open loop transfer function matrix $P(s)$ is square $(m \times m)$ and *all* its zeros are located in the open left half plane, then if a certain sufficient condition is satisfied, the effect of state variable feedback for internal stabilization and arbitrary pole assignment can be accomplished without access to the state variable vector by only output feedback through a *proper and stable* dynamic feedback compensator $C(s)$.

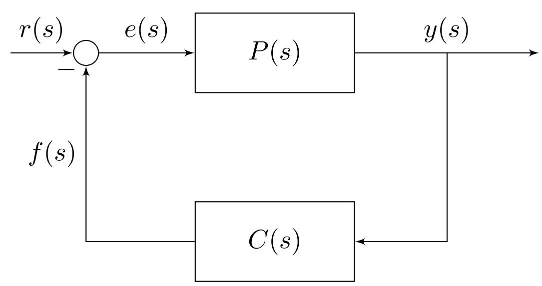

Fig. 1. The closed loop system Σ_c

In the sequel by $\mathbb{R}^{p\times m}$, $\mathbb{R}[s]^{p\times m}$, $\mathbb{R}(s)^{p\times m}$, $\mathbb{R}_{pr}(s)^{p\times m}$, $\mathbb{R}_{pr0}(s)^{p\times m}$ we denote the sets of $p \times m$ matrices with elements in the sets respectively of *reals*, *polynomials, rational*, *proper rational* and *strictly proper rational* functions, in the indeterminate s all with coefficients in the field $\mathbb{R}$ of reals and by $\overline{\mathbb{R}}^{m\times m}$, $\overline{\mathbb{R}}[s]^{m\times m}$, $\overline{\mathbb{R}}(s)^{m\times m}$, $\overline{\mathbb{R}}_{pr}(s)^{m\times m}$, $\overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ we denote the subsets of the corresponding sets whose elements are *non-singular* matrices. The *degree* $\deg T(s)$ of a $T(s) \in \mathbb{R}[s]^{p\times m}$ is defined as *maximum degree* among the degrees of its maximum order non-zero minors so that for a $T(s) \in \overline{\mathbb{R}}[s]^{m\times m}$: $\deg T(s) = \deg(\det T(s))$ and if $P(s) = N_R(s)D_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{p\times m}$, $N_R(s) \in \mathbb{R}[s]^{p\times m}$, $D_R(s) \in \overline{\mathbb{R}}[s]^{m\times m}$, then $\deg N_R(s) \le \deg D_R(s)$ with $\deg N_R(s) < \deg D_R(s)$ iff $P(s) \in \mathbb{R}_{pr0}(s)^{p\times m}$ (Propositions 3.81-3.82 in [3]). For a $T(s) \in \mathbb{R}[s]^{p\times m}$, $\deg_{ci} T(s)$ denotes the degree of the $i-th$ column of $T(s)$, $[T(s)]_c^h \in \mathbb{R}^{p\times m}$ is the *highest column degree coefficient matrix* of $T(s)$ and $c_c(T(s)) = \sum_{i=1}^m \deg_{ci} T(s)$ is the *column complexity* of $T(s)$ and $T(s)$ is *column proper* iff $rank_{\mathbb{R}}[T(s)]_c^h = \min\{p, m\}$ [3]. For a

$P(s) \in \mathbb{R}(s)^{p\times m}$, $\delta_M(P(s))$ denotes the *McMillan degree* and $[P(s)]_{sp} \in \mathbb{R}_{pr0}(s)^{p\times m}$ denotes the *strictly proper part* of $P(s)$. Finally, $\mathbb{C}^- := \{s \in \mathbb{C}, \operatorname{Re} s < 0\}$ is the open left half of the complex plane $\mathbb{C}$ and $\overline{\mathbb{C}}^+ := \{s \in \mathbb{C}, \operatorname{Re} s \geq 0\}$. The rest of the terminology and notation in the sequel is the standard one found in the literature of the "*polynomial matrix approach*" in books like [10],[11],[8],[3],[9].

II. Background and preliminary results

We consider LTI multivariable systems Σ described by input-output *strictly proper* transfer function matrices $P(s) \in \mathbb{R}_{pr0}(s)^{p\times m}$. Two transfer function matrices $(P_1(s), P_2(s)) \in \mathbb{R}_{pr0}(s)^{p\times m} \times \mathbb{R}_{pr0}(s)^{p\times m}$ each one corresponding to systems Σ_1 and Σ_2 are defined as *equivalent under dynamic pre-compensation* [1] (or *dynamically equivalent*) [2]) if

$$P_1(s)T_{C1}(s) = P_2(s), \qquad P_2(s)T_{C2}(s) = P_1(s) \tag{1}$$

for some *biproper* rational matrices $T_{C1}(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ and $T_{C2}(s) := T_{C1}(s)^{-1} \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$. The eqs. in (1) define an equivalence relation $\mathcal{S}$ on $\mathbb{R}_{pr0}(s)^{p\times m}$ which is known as the *equivalence relation under dynamic pre-compensation.* If $(P_1(s), P_2(s))$ are *equivalent under dynamic pre-compensation* we write $(P_1(s), P_2(s)) \in \mathcal{S}$ and for a fixed $P(s) \in \mathbb{R}_{pr0}(s)^{p\times m}$ the $\mathcal{S}$-equivalence class of $P(s)$ is the set $[P(s)]_{\mathcal{S}} := \left\{\hat{P}(s) \in \mathbb{R}_{pr0}(s)^{p\times m} \mid \left(P(s), \hat{P}(s)\right) \in \mathcal{S}\right\}$. Let $\Sigma := \left(A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}\right)$ be a *minimal state space realization* of a $P(s) \in \mathbb{R}_{pr0}(s)^{p\times m}$ and consider the group $\mathcal{G}$ of transformations $\underline{g}$ on Σ defined by

$$\underline{g}\Sigma = \underline{g}(A, B, C) = \left[Q^{-1}(A + BF)Q, Q^{-1}BG, CQ\right]$$

so that each $\underline{g} \in \mathcal{G}$ is represented by a distinct triple of real matrices $\left(Q \in \overline{\mathbb{R}}^{n\times n}, F \in \mathbb{R}^{m\times n}, G \in \overline{\mathbb{R}}^{m\times m}\right)$. Two transfer function matrices $(P_1(s), P_2(s)) \in \mathbb{R}_{pr0}(s)^{p\times m} \times \mathbb{R}_{pr0}(s)^{p\times m}$ with minimal state space realizations

$$\Sigma_i = (A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}, C_i \in \mathbb{R}^{p\times n}),\ i = 1, 2$$

are called *state feedback equivalent* [1],[2] if (i) $\delta_M(P_1(s)) = \delta_M(P_2(s))$ and (ii) there exists an element $\underline{g} \in \mathcal{G}$ such that

$$\left[Q^{-1}(A_1 + B_1F)Q, Q^{-1}B_1G, C_1Q\right] = [A_2, B_2, C_2].$$

The above two requirements define another equivalence relation $\mathcal{F}$ on $\mathbb{R}_{pr0}(s)^{p\times m}$ which is known as the *state feedback group equivalence relation* [2] and if $(P_1(s), P_2(s))$ are *state feedback equivalent* we write $(P_1(s), P_2(s)) \in \mathcal{F}$ and we have

$$P_2(s) = C_2(sI_n - A_2)^{-1}B_2 = C_1(sI_n - A_1 - B_1F)^{-1}B_1G.$$

Considering the biproper rational matrix

$$T_{C1}(s) := F(sI_n - A_1 - B_1F)^{-1}B_1G + G \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$$

then, with $P_1(s) = C_1(sI_n - A_1)^{-1}B_1$ and the *identity*

$$P_1(s)T_{C1}(s) = P_2(s), \tag{2}$$

it becomes evident that if $(P_1(s), P_2(s)) \in \mathcal{F}$, then $P_2(s)$ can be obtained from $P_1(s)$ by cascading $P_1(s)$ with the pre-compensator $T_{C1}(s)$. As $T_{C1}(s)$ is biproper we also have the *identity*

$$P_2(s)T_{C2}(s) = P_1(s) \tag{3}$$

where[1]

$$\begin{aligned} T_{C2}(s) &= T_{C1}(s)^{-1} = -G^{-1}F(sI_n - A_1)^{-1}B_1 + G^{-1} \\ &\in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}. \end{aligned}$$

From the above arguments and eqs. (2), (3) it follows that $(P_1(s), P_2(s)) \in \mathcal{F} \Rightarrow (P_1(s), P_2(s)) \in \mathcal{S}$ but the reverse does not hold true in general. The n. and s. condition(s) under which $(P_1(s), P_2(s)) \in \mathcal{S} \Rightarrow (P_1(s), P_2(s)) \in \mathcal{F}$ was originally examined in [4]. In the sequel, using our notation, we restate these conditions in Proposition 1. To this end let $P(s) \in \mathbb{R}_{pr0}(s)^{p\times m}$ and $\Sigma := \left(A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}\right)$ be a *minimal* state space *realization* of $P(s)$, i.e. let

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

$P(s) = C(sI_n - A)^{-1}B$ and let

$$P(s) = N_R(s)D_R(s)^{-1} \tag{4}$$

be a right coprime MFD of $P(s)$ where

$$N_R(s) = CS(s) \in \mathbb{R}[s]^{p\times m} \tag{5}$$

and $S(s) \in \mathbb{R}[s]^{n\times m}$, $D_R(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ and such that $S(s)D_R(s)^{-1} \in \mathbb{R}_{pr0}(s)^{n\times m}$ is a right coprime MFD of the left coprime MFD: $(sI_n - A)^{-1}B \in \mathbb{R}_{pr0}(s)^{n\times m}$ so that

$$(sI_n - A)^{-1}B = S(s)D_R(s)^{-1}. \tag{6}$$

Consider now Σ under the state feedback control law

$$u(t) = Fx(t) + v(t) \tag{7}$$

where $F \in \mathbb{R}^{m\times n}$ and $v(t)$ a new input vector so that the closed loop system Σ_F under state feedback is described by the state space equations

$$\dot{x}(t) = (A + BF)x(t) + Bv(t), \quad y(t) = Cx(t) \tag{8}$$

and let

$$P_F(s) := C[sI_n - (A + BF)]^{-1}B \in \mathbb{R}_{pr0}(s)^{p\times m} \tag{9}$$

be the transfer function matrix of Σ_F. Let $D_{RF}(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ such that $S(s)D_{RF}(s)^{-1} \in \mathbb{R}_{pr0}(s)^{n\times m}$ is a right coprime MFD of $[sI_n - (A + BF)]^{-1}B \in \mathbb{R}_{pr0}(s)^{n\times m}$ so that

$$[sI_n - (A + BF)]^{-1}B = S(s)D_{RF}(s)^{-1} \tag{10}$$

which in view of (5),(9) implies that

$$P_F(s) = N_R(s)D_{RF}(s)^{-1} \tag{11}$$

[1]If $\left(A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, E \in \mathbb{R}^{m\times m}\right)$ is a minimal realization of a biproper rational matrix $T(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ then $(A - BE^{-1}C \in \mathbb{R}^{n\times n}, BE^{-1} \in \mathbb{R}^{n\times m}, -E^{-1}C \in \mathbb{R}^{m\times n}, E^{-1} \in \mathbb{R}^{m\times m})$ is a minimal realization of the biproper rational matrix $T(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$.

is a right MFD of $P_F(s)$ of Σ_F and $(P(s), P_F(s)) \in \mathcal{F}$. Notice that (10) implies also that

$$(sI_n - A)\,S(s) = B\,[FS(s) + D_{RF}(s)]$$

which in view of (6) can be written as

$$BD_R(s) = B\,[FS(s) + D_{RF}(s)]$$

implying that the (right) closed loop denominator polynomial matrix $D_{RF}(s)$ of $P_F(s)$ is given by [9]

$$D_{RF}(s) = D_R(s) - FS(s). \tag{12}$$

Considering the *biproper* rational matrix

$$T_C(s) := F(sI_n - A - BF)^{-1}B + I_m \in \overline{\mathbb{R}}_{pr}\,(s)^{m\times m},$$

then the *identity*

$$P\,(s)\,T_C(s) = P_F(s) \tag{13}$$

implies also that $(P(s), P_F(s)) \in \mathcal{S}$. In view of (4) and (11) the identity (13) holds true iff

$$T_C(s)^{-1}D_R(s) = D_{RF}(s) \tag{14}$$

Based on the above analysis we can state the next proposition which states that if two transfer function matrices $P_i(s) = N_{Ri}(s)D_{Ri}(s)^{-1}$, $i = 1,2$ are *equivalent under dynamic pre-compensation*, i.e. if $P_1(s)T_C(s) = P_2(s)$ for some some biproper $T_C(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ then they are *state feedback equivalent* iff (15) holds true.

Proposition 1: Let $(P_1(s), P_2(s)) \in \mathbb{R}_{pr0}(s)^{p\times m} \times \mathbb{R}_{pr0}(s)^{p\times m}$ and let $P_i(s) = N_{Ri}(s)D_{Ri}(s)^{-1}$, $i = 1,2$ be respectively right coprime MFDs. If $(P_1(s), P_2\,(s)) \in \mathcal{S}$, i.e. if $P_1(s)T_C(s) = P_2(s)$ for some biproper $T_C(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ then $(P_1(s), P_2\,(s)) \in \mathcal{F}$ iff

$$T_C(s)^{-1}D_{R1}(s) = D_{R2}(s). \tag{15}$$

Our results in the sequel rely on the assumption that the given plant $P(s)$ is *square and non-singular* (see eq. 24) so from now on we assume that $p = m$. The next Proposition 2 will be needed in establishing of our algorithm for the construction of the compensator $C(s)$ with the desired properties. Consider the Euclidean polynomial matrix division [8],[3]

$$D_R(s) = Q\,(s)\,N_R(s) + R_R(s) \tag{16}$$

where $Q\,(s) \in \overline{\mathbb{R}}\,[s]^{m\times m}$ is the *quotient* of $D_R\,(s)$ by $N_R\,(s)$ and $R_R(s) \in \mathbb{R}\,[s]^{m\times m}$ is the *right remainder* and either $R_R(s) = 0$ or $\deg R_R(s) < \deg N_R(s)$ [5] so that (16) can be written as

$$P\,(s)^{-1} = D_R(s)N_R(s)^{-1} = Q\,(s) + R_R(s)N_R(s)^{-1} \tag{17}$$

where $R_R(s)N_R(s)^{-1} \in \mathbb{R}_{pr0}(s)^{m\times m}$. Let $Q\,(s) = \sum_{i=1}^{q} Q_i s^i + Q_0 \in \overline{\mathbb{R}}\,[s]^{m\times m}$, $Q_i \in \mathbb{R}^{m\times m}$ and $Q_+\,(s) := \sum_{i=1}^{q} Q_i s^i = Q\,(s) - Q_0 \in \overline{\mathbb{R}}\,[s]^{m\times m}$ be the *strictly polynomial part* of $P\,(s)^{-1}$ known as the "*first atom*" of $P\,(s)$ [5],[7], $H_R(s) := Q_+\,(s)\,N_R(s)$, and $G_R(s) := Q_0N_R(s) + R_R(s)$ so that (16) can be written as

$$D_R(s) = H_R(s) + G_R(s). \tag{18}$$

Let

$$D_R\,(s) =: [D_{c1}(s)\ \ D_{c2}(s)\ \ \ldots\ \ D_{cm}(s)]\,,$$

$D_{ci}(s) \in \mathbb{R}\,[s]^{m\times 1}$, $\mu_i := \deg D_{ci}(s)$ and $c := c_c\,(D_R\,(s)) = \sum_{i=1}^{m} \mu_i$ and write

$$D_R\,(s) = [D_R\,(s)]_c^h\,diag\,\{s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_m}\} + D_R S(s)$$

where

$$\begin{gathered}
[D_R\,(s)]_c^h \in \mathbb{R}^{m\times m},\\
S(s) = blockdiag\,\{S_1(s), S_2(s), \ldots, S_m(s)\} \in \mathbb{R}\,[s]^{c\times m},\\
S_i(s) := \begin{bmatrix}1 & s & \ldots & s^{\mu_i - 1}\end{bmatrix}^{\mathsf{T}} \in \mathbb{R}\,[s]^{\mu_i\times 1},\\
[D_R\,(s)]_c^h =: [D_1\ \ D_2\ \ \ldots\ \ D_m]\,,\ D_i \in \mathbb{R}^{m\times 1},\\
D_R =: [D_{R1}\ \ D_{R2}\ \ \ldots\ \ D_{Rm}] \in \mathbb{R}\,[s]^{m\times c},\\
D_{Ri} = \begin{bmatrix}D_{i0} & D_{i1} & \ldots & D_{i,\mu_i - 1}\end{bmatrix} \in \mathbb{R}^{m\times\mu_i},\ D_{ij} \in \mathbb{R}^{m\times 1},
\end{gathered} \tag{19}$$

then

$$D_{ci}(s) = D_i s^{\mu_i} + D_{Ri}S_i(s),\ i = 1,2,\ldots,m.$$

Similarly write

$$H_R\,(s) =: [H_{c1}(s)\ \ H_{c2}(s)\ \ \ldots\ \ H_{cm}(s)]\,,$$

$H_{ci}(s) \in \mathbb{R}\,[s]^{m\times 1}$, $\deg H_{ci}(s) = \mu_i$ so that

$$\begin{gathered}
H_R\,(s) = [H_R\,(s)]_c^h\,diag\,[s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_m}] + H_R S(s),\\
[H_R\,(s)]_c^h =: [H_1\ \ H_2\ \ \ldots\ \ H_m]\,,\ H_i \in \mathbb{R}^{m\times 1},\\
H_R =: [H_{R1}\ \ H_{R2}\ \ \ldots\ \ H_{Rm}] \in \mathbb{R}\,[s]^{m\times c},\\
H_{Ri} = \begin{bmatrix}\mathbf{0}_{m\times 1} & H_{i1} & \ldots & H_{i,\mu_i - 1}\end{bmatrix} \in \mathbb{R}^{m\times\mu_i},\ H_{ij} \in \mathbb{R}^{m\times 1},
\end{gathered} \tag{20}$$

where the first column of H_{Ri} is zero due to the fact that $Q_+\,(s)$ is *strictly polynomial* and

$$H_{ci}(s) = H_i s^{\mu_i} + H_{Ri}S_i(s),\ i = 1,2,\ldots,m,$$

Finally write

$$G_R\,(s) =: [G_{c1}(s)\ \ G_{c2}(s)\ \ \ldots\ \ G_{cm}(s)]\,,$$

$\deg G_{ci}(s) =: \rho_i < \mu_i$,

$$\begin{gathered}
G_R\,(s) = G_R S(s),\\
G_R =: [G_{R1}\ \ G_{R2}\ \ \ldots\ \ G_{Rm}] \in \mathbb{R}\,[s]^{m\times c},\\
G_{Ri} = \begin{bmatrix}G_{i0} & G_{i1} & \ldots & G_{i,\rho_i} & \mathbf{0}_{m\times(\mu_i - \rho_i - 1)}\end{bmatrix} \in \mathbb{R}^{m\times\mu_i},\\
G_{ij} \in \mathbb{R}^{m\times 1},
\end{gathered} \tag{21}$$

$$G_{ci}(s) = G_{Ri}S_i(s),\ i = 1,2,\ldots,m$$

so that from (18)

$$D_i s^{\mu_i} + D_{Ri}S_i(s) = H_i s^{\mu_i} + (H_{Ri} + G_{Ri})\,S_i(s), \tag{22}$$

$$i = 1,2,\ldots,m.$$

By combining (19), (20) and (21) with (22) we obtain

Proposition 2: In the Euclidean division $D_R(s) = Q\,(s)\,N_R(s) + R_R(s) = H_R(s) + G_R(s)$ it holds that

$$[D_R\,(s)]_c^h = [H_R\,(s)]_c^h\,,$$

$$D_{il} = H_{il},\ l = \rho_i + 1, \rho_i + 2, \ldots, \mu_i - 1.$$

III. Proper feedback compensators equivalent to state variable feedback

Consider now a stabilizable system Σ with transfer function matrix $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and let

$$P_C(s) = P(s)[I_m + C(s)P(s)]^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \quad (23)$$

the transfer function matrix of the closed loop system Σ_c in figure 1, obtained by output feedback through a *proper* compensator $C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ in the feedback path so that $\delta_M(P_C(s)) \le \delta_M(P(s)) + \delta_M(C(s))$ with equality holding iff there are no pole-zero cancellations in the product $C(s)P(s)$. Writing (23) as

$$P_C(s)^{-1} = P(s)^{-1} + C(s) \quad (24)$$

gives rise to the *dynamic feedback equivalence relation* $\mathcal{R}$ of $P(s)$ which was examined in [5] according to which $(P_1(s), P_2(s)) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \times \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ are called *dynamically feedback equivalent* (via $C(s)$) if $P_1(s)^{-1} - P_2(s)^{-1} =: C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$. In such a case we denote this fact by writing $(P_1(s), P_2(s)) \in \mathcal{R}$ and the *dynamic feedback equivalence class* of a $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ of the *dynamic feedback equivalence relation* $\mathcal{R}$ by $[P(s)]_{\mathcal{R}}$. It has been shown (*Proposition* 5 in [5]) that if $f : \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \to \overline{\mathbb{R}}[s]^{m\times m}$ is the map $P(s) \mapsto Q_+(s)$ then for a $P_1(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$: $(P(s), P_1(s)) \in \mathcal{R} \Leftrightarrow fP(s) = fP_1(s)$, i.e. the strictly polynomial part: $Q_+(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ of $P(s)^{-1}$ is a complete invariant of $[P(s)]_{\mathcal{R}}$. Let us now assume that there exists a proper compensator $C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ such that the closed loop transfer function matrix $P_C(s)$ is equal to the closed loop transfer function matrix $P_F(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ obtained from $P(s)$ via the state variable feedback control law in (7), i.e. let us assume that there exists a proper feedback compensator $C(s)$ such that

$$P_C(s) = P_F(s) \quad (25)$$

so that $\delta_M(P_C(s)) = \delta_M(P_F(s)) = \delta_M(P(s))$. Then from (24) and (25) such a compensator can be written as

$$C(s) = P_F(s)^{-1} - P(s)^{-1}.$$

Let $T_C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ be the "open loop precompensator" of $P(s)$ which is equivalent to the state feedback law in (7) giving rise to $P_C(s)$ satisfying (25), so that eq. (13) can be written as

$$P(s)T_C(s) = P_C(s) = P_F(s). \quad (26)$$

Then from (23) the precompensator $T_C(s)$ in (26) will be given by $T_C(s) = [I_m + C(s)P(s)]^{-1}$ so that

$$T_C(s)^{-1} = I_m + C(s)P(s). \quad (27)$$

Multiplying (27) on the right by $D_R(s)$ and in view of (14), (27) can be written as

$$T_C(s)^{-1}D_R(s) \overset{(14)}{=} D_{RF}(s) = [I_m + C(s)P(s)]\, D_R(s)$$

so that $I_m + C(s)P(s) = D_{RF}(s)D_R(s)^{-1}$ or equivalently that $C(s)P(s) = D_{RF}(s)D_R(s)^{-1} - I_m$ which, since $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and $P(s)^{-1} = D_R(s)N_R(s)^{-1}$, gives finally that the feedback compensator $C(s)$ giving rise to (25) is

$$\begin{aligned} C(s) &= \left[D_{RF}(s)D_R(s)^{-1} - I_m\right] P(s)^{-1} \\ &= D_{RF}(s)N_R(s)^{-1} - D_R(s)N_R(s)^{-1} \\ &= \left[D_{RF}(s) - D_R(s)\right] N_R(s)^{-1} \qquad (28) \\ &\overset{(12)}{=} -FS(s)N_R(s)^{-1} =: Y(s)X(s)^{-1}. \end{aligned}$$

We define the output feedback (and possibly non-proper) dynamic compensator $C(s)$ in (28) as the "*State Feedback Dynamic (SFD) compensator*". An obvious question one can pose now is under what conditions the *SFD compensator* $C(s)$ in (28) (i) is a *proper* rational matrix and (ii) gives rise to an internally stable closed loop system Σ_c with arbitrary poles. Let $P(s) = D_L(s)^{-1}N_L(s)$ be a left coprime MFD of $P(s)$. Then, if the feedback compensator in (28) is proper, the *characteristic polynomial matrix* of the closed loop system Σ_c is

$$\begin{aligned} & D_L(s)X(s) + N_L(s)Y(s) \\ =\ & D_L(s)N_R(s) + N_L(s)\left[D_{RF}(s) - D_R(s)\right] \\ =\ & N_L(s)D_{RF}(s) \qquad (29) \end{aligned}$$

and we have

Proposition 3: With the SFD compensator $C(s)$ in (28), the closed loop system Σ_c with transfer function matrix $P_C(s) = N_R(s)D_{RF}(s)^{-1}$ is *internally stable* iff the characteristic polynomial matrix of Σ_c in (29) or equivalently the polynomial matrices $N_L(s)$, $D_{RF}(s)$ have all their zeros in $\mathbb{C}^-$.

As it will be shown in the sequel (Theorem 7 below), under a certain sufficient condition, the desired denominator $D_{RF}(s)$ of $P_C(s)$ can be chosen having all its zeros in $\mathbb{C}^-$ via the choice of a *proper* SFD compensator as in (28), and thus we can state

Proposition 4: The class $\mathcal{P}$ of transfer function matrices $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ that under the SFD compensator in (28) give rise to internally stable closed loop systems Σ_c with transfer function matrix $P_C(s) = P_F(s) \in [P(s)]_{\mathcal{F}}$ is the class of transfer function matrices with no zeros in $\overline{\mathbb{C}}^+$, i.e. such that $\det N_L(s) \neq 0$, $\forall s \in \overline{\mathbb{C}}^+$ or equivalently $\det N_R(s) \neq 0$, $\forall s \in \overline{\mathbb{C}}^+$.

The next theorem gives the necessary and sufficient condition for the SFD compensator $C(s)$ in (28) to be a *proper* rational matrix.

Theorem 5: Let Σ be a LTI stabilizable multivariable system with transfer function matrix $P(s) = N_R(s)D_R(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$. Then the SFD compensator $C(s)$ in (28), which gives rise to the closed loop system Σ_c in (8) with transfer function matrix $P_C(s) = P_F(s) = N_R(s)D_{RF}(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$, is *proper* iff $(P(s), P_F(s)) \in \mathcal{R}$. Equivalently $C(s) = P_F(s)^{-1} - P(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m} \Leftrightarrow fP_F(s) = fP(s) = Q_+(s)$.

Proof: ($\Rightarrow$) Let $C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ but contrary to the conclusion let us assume that $fP(s) = Q_+(s) \neq$

$Q_+^F(s) := fP_F(s)$. Then due to the fact that $\mathbf{0}_{m\times m} \neq Q_+^F(s) - Q_+(s) \in \mathbb{R}[s]^{m\times m}$,

$$\begin{aligned} C(s) &= P_F(s)^{-1} - P(s)^{-1} \\ &= \left[Q_+^F(s) - Q_+(s)\right] \\ &\quad + \left[Q_0^F - Q_0\right] + \left[R_{RF}(s) - R_R(s)\right] N_R(s)^{-1} \\ &\notin \mathbb{R}_{pr}(s)^{m\times m} \end{aligned}$$

which contradicts the assumption.

($\Leftarrow$) From (17)

$$\begin{aligned} P(s)^{-1} &= D_R(s)N_R(s)^{-1} \\ &= Q_+(s) + Q_0 + R_R(s)N_R(s)^{-1}. \end{aligned}$$

Consider now the polynomial matrix Euclidean division $D_{RF}(s) = Q^F(s)N_R(s) + R_{RF}(s)$ where due to the assumption: $Q^F(s) = Q_+(s) + Q_0^F \in \overline{\mathbb{R}}[s]^{m\times m}$ is the quotient, $Q_0^F \in \mathbb{R}^{m\times m}$ and $R_{RF}(s) \in \mathbb{R}[s]^{m\times m}$ is the right remainder so that $R_{RF}(s)N_R(s)^{-1} \in \mathbb{R}_{pr0}(s)^{m\times m}$ or equivalently

$$\begin{aligned} D_{RF}(s) &= Q_+(s)N_R(s) + Q_0^F N_R(s) + R_{RF}(s) &(30)\\ &= H_R(s) + G_{RF}(s) &(31) \end{aligned}$$

where $G_{RF}(s) := Q_0^F N_R(s) + R_{RF}(s)$, from which it follows that

$$\begin{aligned} P_F(s)^{-1} &= D_{RF}(s)N_R(s)^{-1} \\ &= Q_+(s) + Q_0^F + R_{RF}(s)N_R(s)^{-1} \end{aligned}$$

so that from (28)

$$C(s) = P_F(s)^{-1} - P(s)^{-1}$$

$$= \left[Q_0^F - Q_0\right] + \left[R_{RF}(s) - R_R(s)\right] N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}.$$

■

In the sequel we will make use of the following lemma [6].

Lemma 6: Let $A(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ and write it as $A(s) = [A(s)]_c^h \, diag\{s^{r_1}, s^{r_2}, ..., s^{r_m}\} + A\hat{S}(s)$, $\hat{S}(s) = blockdiag\left\{\hat{S}_1(s), \hat{S}_2(s), ..., \hat{S}_m(s)\right\} \in \mathbb{R}[s]^{\hat{c}\times m}$, $\hat{c} := c_c(A(s))$, $\hat{S}_i(s) := \left[1 \;\; s \;\; ... \;\; s^{r_i-1}\right]^{\mathsf{T}} \in \mathbb{R}[s]^{r_i\times 1}$, $[A(s)]_c^h =: [A_{1,r_1} \; ... \; A_{m,r_m}]$, $A_{i,r_i} \in \mathbb{R}^{m\times 1}$, $A =: [A_1 \; ... \; A_m]$, $A_i \in \mathbb{R}^{m\times r_i}$, $A_i = [A_{i0} \;\; A_{i1} \;\; ... \;\; A_{i,r_i-1}]$, $A_{ij} \in \mathbb{R}^{m\times 1}$, then if $\det A(s) = \sum_{k=0}^n a_k s^k$, $n = \deg A(s)$

$$a_k = \sum_{i_1+...+i_m=k} \det\left[A_{1,i_1} \;\; ... \;\; A_{m,i_m}\right].$$

The next Theorem is our main result and gives a sufficient condition for the *SFD compensator* $C(s)$ in (28) to be *proper, internally stabilizing and denominator (pole) assigning*.

Theorem 7: Let Σ be a LTI stabilizable multivariable system with transfer function matrix $P(s) = N_R(s)D_R(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ with $D_R(s)$ column proper. If $P(s) \in \mathcal{P}$ and $D_{RF}(s)$ is the closed loop right denominator matrix in (12) obtained by a state feedback law as in (7) and having arbitrary desired zeros, then the *SFD compensator* $C(s)$ in (28) is *proper* and gives rise to an *internally stable* closed loop system Σ_c with transfer function matrix $P_C(s) = P_F(s) := N_R(s)D_{RF}(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ if

$$\nu_i := \deg_{ci} N_R(s) = \deg_{ci} D_R(s) - 1 =: \mu_i - 1 \qquad (32)$$

for at least one $i \in [1, m]$ and $\deg N_R(s) \geq \frac{n}{m} - m$.

Proof: According to Theorem 5 the desired denominator in (31) must have the form

$$D_{RF}(s) = Q_+(s)N_R(s) + G_{RF}(s)$$

where $G_{RF}(s)$ is yet unknown. From Proposition 2 it follows that

$$[D_{RF}(s)]_c^h = [H_R(s)]_c^h = [D_R(s)]_c^h.$$

Let $U(s) \in \overline{\mathbb{R}}[s]^{\;m\times m}$ and unimodular such that $\overline{N}_R(s) := N_R(s)U(s)$ is column proper. Let $\overline{G}_{RF}(s) \in \mathbb{R}[s]^{\;m\times m}$ with $\deg_{ci} \overline{G}_{RF}(s) = \deg_{ci} \overline{N}_R(s) =: \overline{\nu}_i$,

$$\overline{G}_{RF}(s) := \overline{G}_{RF}\overline{S}(s)$$

where $\overline{G}_{RF} = [x_{jk}] \in \mathbb{R}^{m\times\tau}$, $\tau := \sum_{i=1}^m (\overline{\nu}_i + 1)$ is a symbolic matrix of $m\tau$ unknowns x_{jk},

$$\overline{S}(s) = blockdiag\left\{\overline{S}_1(s), \overline{S}_2(s), ..., \overline{S}_m(s)\right\} \in \mathbb{R}[s]^{\tau\times m},$$

$$\overline{S}_i(s) := \left[1 \;\; s \;\; ... \;\; s^{\overline{\nu}_i}\right]^{\mathsf{T}} \in \mathbb{R}[s]^{(\overline{\nu}_i+1)\times 1}$$

so that $\overline{G}_{RF}(s)\overline{N}_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ [3]. Then

$$\begin{aligned} \overline{G}_{RF}(s)\overline{N}_R(s)^{-1} &= \overline{G}_{RF}(s)U(s)^{-1}N_R(s)^{-1} \\ &= G_{RF}(s)N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m} \end{aligned}$$

with $G_{RF}(s) := \overline{G}_{RF}(s)U(s)^{-1}$ and the *SFD compensator* $C(s)$ in (28) is

$$\begin{aligned} C(s) &= \left[D_{RF}(s) - D_R(s)\right] N_R(s)^{-1} \\ &= \left[G_{RF}(s) - G_R(s)\right] N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}. \end{aligned}$$

Write

$$D_{RF}(s) = [D_R(s)]_c^h \, diag\{s^{\mu_1}, s^{\mu_2}, ..., s^{\mu_m}\} + D_{RF}S(s), \qquad (33)$$

$$D_{RF} =: [D_{RF1} \;\; D_{RF2} \;\; ... \;\; D_{RFm}] \in \mathbb{R}^{m\times c},$$

$$D_{RFi} = \left[D_{i0}^F \;\; D_{i1}^F \;\; ... \;\; D_{i,\mu_i-1}^F\right] \in \mathbb{R}^{m\times\mu_i}, \; D_{ij}^F \in \mathbb{R}^{m\times 1},$$

$$G_{RF}(s) = G_{RF}S(s),$$

$$G_{RF} =: [G_{RF1} \;\; G_{RF2} \;\; ... \;\; G_{RFm}] \in \mathbb{R}[s]^{m\times c},$$

$$G_{RFi} = \left[G_{i0}^F \;\; G_{i1}^F \;\; ... \;\; G_{i,\rho_i}^F \;\; \mathbf{0}_{m\times(\mu_i-\rho_i-1)}\right] \in \mathbb{R}^{m\times\mu_i},$$

$$G_{ij}^F \in \mathbb{R}^{m\times 1},$$

then by analogy to (22) in Proposition 2 we have

$$D_i s^{\mu_i} + D_{RFi}S_i(s) = H_i s^{\mu_i} + (H_{Ri} + G_{RFi}) S_i(s),$$

$$i = 1, 2, ..., m$$

from which it follows that

$$D_{RFi} = H_{Ri} + G_{RFi}, \; i = 1, 2, ..., m \Rightarrow$$

$$\left[D_{i0}^F \;\; D_{i1}^F \;\; ... \;\; D_{i,\nu_i}^F \;\; D_{i,\nu_i+1}^F \;\; ... \;\; D_{i,\mu_i-1}^F\right] =$$

$$\left[G_{i0}^F \;\; H_{i1}+G_{i1}^F \;\; \ldots \;\; H_{i,\nu_i}+G_{i,\nu_i}^F \;\; H_{i,\nu_i+1} \;\; \ldots \;\; H_{i,\mu_i-1}\right].$$

If (32) holds true for at least one $i \in [1,m]$, then $G_{RFi} = \left[G_{i0}^F \;\; G_{i1}^F \;\; \ldots \;\; G_{i,\mu_i-1}^F\right]$ for these particular indices i. Consider now the characteristic polynomial $\det D_{RF}(s) = \sum_{k=0}^{n} \alpha_k s^k$ of the closed loop system Σ_c then from Lemma 6

$$\alpha_n = \det\left[D_1 \;\; \ldots \;\; D_m\right] = \det\left[D_R(s)\right]_c^h,$$

$$\alpha_k = \sum_{i_1+\ldots+i_m=k} \det\left[D_{1,i_1}^F \;\; \ldots \;\; D_{m,i_m}^F\right]$$

$$= \sum_{i_1+\ldots+i_m=k} \det\left[H_{1,i_1}+G_{1,i_1}^F \;\; \ldots \;\; H_{m,i_m}+G_{m,i_m}^F\right],$$

$$k = 0,\ldots,n-1$$

from which it follows that

$$\begin{aligned} \alpha_{n-1} &= \sum_{i_1+\ldots+i_m=n-1} \det\left[D_{1,i_1}^F \;\; \ldots \;\; D_{m,i_m}^F\right] \\ &= \det\left[D_{1,\mu_1-1}^F \;\; \ldots \;\; D_m\right]+\ldots \\ &\quad +\det\left[D_1 \;\; \ldots \;\; D_{m,\mu_m-1}^F\right]. \end{aligned}$$

For the indices i that (32) holds true we have that

$$\det\left[D_1 \;\; \ldots \;\; D_{i,\mu_i-1}^F \;\; \ldots \;\; D_m\right] =$$
$$\det\left[D_1 \;\; \ldots \;\; H_{i,\mu_{i-1}}+G_{i,\mu_i-1}^F \;\; \ldots \;\; D_m\right]$$

where, due to the fact that the columns G_{i,μ_i-1}^F are symbolic, the coefficient α_{n-1} is also symbolic. Similarly it can be shown that the coefficients $\alpha_k, \; k=0,\ldots,n-2$ are also symbolic. Thus any closed loop characteristic polynomial $\alpha(s) = \det\left[D_R(s)\right]_c^h \hat{\alpha}(s)$, where $\hat{\alpha}(s) \in \mathbb{R}[s]$ is monic and has desired zeros, can be obtained by an appropriate choice of $\overline{G}_{RF}$. In turn the choice of $\overline{G}_{RF}$ can be obtained by solving the system $\det D_{RF}(s) = \alpha(s)$ which involves n equations and r unknowns, where

$$\begin{aligned} r &= m\tau = m\textstyle\sum_{i=1}^{m}\left(\overline{\nu}_i+1\right) = m\left(\sum_{i=1}^{m}\overline{\nu}_i+m\right) \\ &= m\left(\deg \overline{N}_R(s)+m\right) = m\left(\deg N_R(s)+m\right) \end{aligned}$$

is the number of the elements x_{jk} of $\overline{G}_{RF}$. If $\deg N_R(s) \geq \frac{n}{m}-m$ then $r = m\left(\deg N_R(s)+m\right) \geq n$ which implies that the system of equations has always a solution by an appropriate choice of the elements x_{jk} of $\overline{G}_{RF}$. ■

Remark 8: If (32) holds true $\forall i = 1,2,\ldots,m$ then, from (33) and for any Q_0^F in (30), we have that $\left[D_{RF}(s)\right]_c^h = \left[D_R(s)\right]_c^h$ and D_{RF} can be chosen arbitrarily. In such a case we choose $D_{RF}(s) = \left[D_R(s)\right]_c^h \hat{D}_{RF}(s)$, where $\hat{D}_{RF}(s) = diag\left\{d_1^F(s), d_2^F(s), \ldots, d_m^F(s)\right\}$, $d_i^F(s) \in \mathbb{R}[s]$ with $\deg d_i^F(s) = \mu_i$, $i = 1,2,\ldots,m$ monic with desired zeros in $\mathbb{C}^-$ and the *SFD compensator* $C(s)$ can be computed directly from (28).

IV. Computational algorithm for SFD compensators

Given a LTI stabilizable multivariable system Σ characterized by a $m \times m$ non-singular strictly proper transfer function matrix $P(s) \in \mathcal{P}$, then the algorithm to compute a *proper* dynamic compensator $C(s)$ resulting to a closed loop feedback system Σ_c as in figure 1 which is internally stable with a closed loop transfer function matrix $P_C(s)$ which can be obtained from a minimal state space realization of the open loop transfer function matrix $P(s) = N_R(s)D_R(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ with $D_R(s)$ column proper and by the action of state variable feedback is:

1) If $\nu_i := \deg_{ci} N_R(s) = \deg_{ci} D_R(s)-1 =: \mu_i - 1$, $\forall i = 1,2,\ldots,m$, then
2) Choose $\hat{D}_{RF}(s) = diag\left\{d_1^F(s), d_2^F(s), \ldots, d_m^F(s)\right\}$, $d_i^F(s) \in \mathbb{R}[s]$ with $\deg d_i^F(s) = \mu_i$, $i = 1,2,\ldots,m$ monic with desired zeros in $\mathbb{C}^-$ and compute $D_{RF}(s) = \left[D_R(s)\right]_c^h \hat{D}_{RF}(s)$, else
3) If $\nu_i = \deg_{ci} N_R(s) = \deg_{ci} D_R(s)-1 = \mu_i - 1$ for at least one $i \in [1,m]$ and $\deg N_R(s) \geq \frac{n}{m}-m$, then
4) Determine $U(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ and unimodular such that $\overline{N}_R(s) = N_R(s)U(s)$ is column proper and compute $U(s)^{-1} \in \overline{\mathbb{R}}[s]^{m\times m}$, $\overline{\nu}_i := \deg_{ci} \overline{N}_R(s)$.
5) Define $\overline{G}_{RF}(s) := \overline{G}_{RF}\overline{S}(s)$, where $\overline{G}_{RF} = \left[x_{jk}\right] \in \mathbb{R}^{m\times\tau}$, $\tau := \sum_{i=1}^{m}\left(\overline{\nu}_i+1\right)$ is symbolic and $\overline{S}(s) = blockdiag\left\{\overline{S}_1(s), \overline{S}_2(s), \ldots, \overline{S}_m(s)\right\} \in \mathbb{R}[s]^{\tau\times m}$, $\overline{S}_i(s) := \left[1 \;\; s \;\; \ldots \;\; s^{\overline{\nu}_i}\right]^\intercal \in \mathbb{R}[s]^{(\overline{\nu}_i+1)\times 1}$ and compute $G_{RF}(s) = \overline{G}_{RF}(s)U(s)^{-1}$.
6) Compute $P(s)^{-1} = Q(s)+\left[P(s)^{-1}\right]_{sp}$ and $Q_+(s) = Q(s)-Q_0$.
7) Compute symbolic values for $D_{RF}(s) = Q_+(s)N_R(s)+G_{RF}(s)$ and $\det D_{RF}(s)$.
8) Define $\hat{\alpha}(s) \in \mathbb{R}[s]$ with $\deg\hat{\alpha}(s) = \deg D_R(s)$, monic with desired zeros in $\mathbb{C}^-$ and compute $\alpha(s) = \det\left[D_R(s)\right]_c^h \hat{\alpha}(s)$. By equating coefficients of equal degree terms in $\det D_{RF}(s) = \alpha(s)$ obtain a numerical solution for the unknown values of $\overline{G}_{RF}$ and compute $D_{RF}(s) = Q_+(s)N_R(s)+G_{RF}(s)$ by substituting $\overline{G}_{RF}$, else end.
9) Compute the *SFD compensator* $C(s) = \left[D_{RF}(s)-D_R(s)\right]N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ and $P_C(s) = N_R(s)D_{RF}(s)^{-1}$.

Example 9: Let

$$\begin{aligned} P(s) &= \begin{bmatrix} \frac{3s^3+6s^2+4s+4}{6s^4} & \frac{2(s+1)}{3s^4} \\ \frac{(s+1)^2}{3s^3} & \frac{(s+1)^2}{3s^3} \end{bmatrix} = N_R(s)D_R(s)^{-1} \\ &= \begin{bmatrix} s+2 & 1 \\ 0 & (s+1)^2 \end{bmatrix}\begin{bmatrix} 2s^2 & -2s \\ -2s^2 & 3s^3+2s \end{bmatrix}^{-1} \end{aligned}$$

so that $m=2$,

$$\det\left[D_R(s)\right]_c^h = \det\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix} = 6 \neq 0$$

and $D_R(s)$ is column proper, $\det N_R(s) = (s+2)(s+1)^2$

therefore $P(s) \in \mathcal{P}$. Now

$$\begin{aligned} \nu_1 &= \deg_{c1} N_R(s) = 1 = \deg_{c1} D_R(s) - 1 = \mu_1 - 1, \\ \nu_2 &= \deg_{c2} N_R(s) = 2 = \deg_{c2} D_R(s) - 1 = \mu_2 - 1 \end{aligned}$$

and thus we choose

$$\hat{D}_{RF}(s) = \begin{bmatrix} (s+3)^2 & 0 \\ 0 & (s+4)^3 \end{bmatrix}$$

and compute

$$\begin{aligned} D_{RF}(s) &= [D_R(s)]_c^h \hat{D}_{RF}(s) \\ &= \begin{bmatrix} 2(s+3)^2 & 0 \\ -2(s+3)^2 & 3(s+4)^3 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} C(s) &= [D_{RF}(s) - D_R(s)] N_R(s)^{-1} \\ &= \begin{bmatrix} \frac{6(2s+3)}{s+2} & \frac{2\left(s^2-4s-9\right)}{(s+2)(s+1)^2} \\ -\frac{6(2s+3)}{s+2} & \frac{2\left(18s^3+107s^2+244s+201\right)}{(s+2)(s+1)^2} \end{bmatrix}, \end{aligned}$$

$$P_C(s) = \begin{bmatrix} \frac{3s^4+42s^3+218s^2+492s+402}{6(s+3)^2(s+4)^3} & \frac{1}{3(s+4)^3} \\ \frac{(s+1)^2}{3(s+4)^3} & \frac{(s+1)^2}{3(s+4)^3} \end{bmatrix}.$$

V. Conclusions

In this paper we examined relations between denominator assigning proper compensators in the feedback path of linear, time invariant multivariable systems and pole assignment by state variable feedback. Through these results we examined the problem of existence and computation of proper compensators $C(s)$ resulting to a closed loop feedback system Σ_c, which is internally stable and is characterized by a closed loop transfer function matrix $P_C(s)$ which is equal to the transfer function matrix $P_F(s)$ that can be obtained by state feedback on a minimal state space realization of $P(s)$. It was shown that if all the zeros of $P(s)$ are located in the open left half plane and a certain sufficient condition is satisfied, then the effect of state variable feedback for internal stabilization and arbitrary pole assignment can be accomplished without access to the state variable vector by only output feedback through a *proper and stable* dynamic feedback compensator $C(s)$.

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